Body Motion, Early Algebra and the Colours of Abstraction

Ricardo Nemirovsky[1] Francesca Ferrara[2] Giulia Ferrari[2] and Natividad Adamuz-Povedano[3]

[1] Manchester Metropolitan University, ESRI, Manchester, UK; r.nemirovsky@mmu.ac.uk

[2] Università di Torino, Italy; francesca.ferrara@unito.it, giulia.ferrari@unito.it

[3] Universidad de Córdoba, Spain; nadamuz@uco.es

Abstract

This paper focuses on the emergence of abstraction through the use of a new kind of motion detector — WiiGraph — with 11-year old children. In the selected episodes, the children used the sensor to create three simultaneous graphs of position vs. time: two graphs for the motion of each hand and a third one corresponding to their difference. They explored relationships that can be ascribed to an equation of the type A – B = C. We examine the notion of abstraction on its own, without assuming a dualism abstract-concrete according to which more of one is less of the other. We propose two distinct paths for the attainment of abstraction, one focused on working with unknowns lacking sensible qualities, and another that involves navigating a surplus of sensible qualities. The work described in this paper belongs to early algebra, we suggest, because it involves the elementary symbolic treatment of unknowns and generals. We also situate it within the context of a specific family of mathematical instruments that involve kinaesthesia and the use of sensors.

Introduction

Learning mathematics is often seen as a progression or movement from the concrete to the abstract. This progression amounts to a passage across emphases, from the sensible to the intelligible. An archetypal example is that of the straight line. Out of countless acts of drawing, touching straight edges, tracing on the sand, or using tools, a sense grows for physical straightness. There is still a major gap between the latter and a geometric straight line involving a massive drawing out of sensible qualities, such as colour, length, material, and thickness, to envision an entity that is intelligible but not sensible. Hence abstraction is depicted as a subtractive process, along which more and more qualities are taken out until a spectral remainder is left that is not amenable to being touched, seen, or heard, and is devoid of causal powers, whose presence is only indirectly evoked by diagrams and formulae. Numerous researchers in mathematics education have questioned this traditional image for the attainment of abstraction (Clements, 2000; Dreyfus, 2014; Hershkowitz, Schwarz, & Dreyfus, 2001; Noss, Hoyles, & Pozzi, 2002; Roth & Hwang, 2006). Wilensky (1991) argued that ideas are abstract or concrete depending on how thinkers relate to them. Someone practiced with linear equations, for instance, might sense a concreteness in them that is unavailable to someone unfamiliar with them, for whom they are abstract. From this point of view abstraction is a deficient mode and the learning of mathematics is rather a progression from the abstract to the concrete, which he called "concretion." Clements (2000) pointed out that the roots of the word "concrete" lie in the idea of growing together and introduces two kinds of concrete knowledge: sensory-concrete and integrated-concrete. The first implies the use of sensory material in the process of sense making; the second combines ideas towards a new structure and implicates physical and abstract knowledge. Noss et al. (2002) introduced the notion of "situated abstraction" seeking to describe how a conceptualisation of mathematical knowledge can simultaneously implicate both, the specificity of a situation and the generality of an abstraction, in a way that these two aspects are interwoven and can feed one another. Hershkowitz et al. (2001) proposed to think of "abstraction in context" to avoid a description of abstraction as some type of decontextualization — akin to our radical "subtraction process" — which they find in most cognitivist approaches. Abstraction in context is an activity or process of reorganization of previous mathematical knowledge into new mathematical structures that incorporate the context motivating it. These researchers followed Davydov (1990) epistemological theory considering a dialectical connection between abstract and concrete. According to Davydov, there are two types of abstraction: empirical and theoretical. An empirical abstraction involves the isolation of a certain perceivable quality common to a set of instances; a theoretical abstraction is organized around theoretical models interrelating unperceivable features participating in the genesis or formation of members of a certain class. Roth and Hwang (2006) analysed a "think aloud" interview with an ecologist as he made sense of a graph he had not used previously. On the basis of a microanalysis of utterances and gestures they conclude that "rather than being a movement from concrete to abstract or from abstract to concrete, development occurs in a movement that appears to be simultaneously from concrete to abstract and from abstract to concrete." (p. 318). Coles and Sinclair (2018) critique the assumption that learning should begin with the concrete and familiar while abstraction arrives later. They argue for a relational view that challenges what is basic and meaningful in the context of number learning.

Drawing on some of the work reviewed above, such as Hershkowitz et al. (2001) and Clements (2000), we think it is important to examine the notions of abstract and concrete on their own, instead of secluding them into a confining dualism, according to which more of one is less of the other. In this paper, borrowing from Peircean philosophical vision about generals, we propose a dynamic vision of abstraction — on its own conceptual distinctiveness — as part of the processes of sense making. We focus on early algebra striving to: 1) articulate a framework contextualising the dynamics of the abstract, in ways that break free from a dualistic co-determination of concreteness, 2) trace two distinct paths for the attainment of abstraction, which we call paths of white and black light, and 3) develop a case study for the pursuit of abstraction along a path of white light. We introduce each of these in the next section.

Theoretical Framework

1. Sense and Reference

In 1892 Frege (1980) published one of his most influential papers whose translated title is "On Sense and Reference". In this paper he illustrated the distinction with the famous example of Venus: the planet is the evening star (i.e. Venus becomes visible first after the sunset) and morning star (i.e. Venus becomes visible before sunrise); the idea being that the same referent, Venus, can be referred through different senses, such as the 'morning star' and the 'evening star'. Enacting different senses for the same reference is pervasive in linguistic interactions. We easily understand, for instance, of someone called Mary, that "Mary is a baseball player" and "Mary is a dedicated student" convey two different senses for the same person. In general, a proposition pinpoints a referent by means of a certain sense, chosen among multiple senses that are possible for it. This is true of algebraic propositions as well. For example, in reference to the following quadratic equation:

$$y = 2x^2 + x + 1 \qquad (1.1)$$

it can be said: "Equation (1.1) has complex roots" and "Equation (1.1) describes a motion with constant acceleration". These two different senses of the same equation can be more or less significant depending on the situation the speaker is grappling with. To capture this active texture of situation and utterance, it is useful to adopt Deleuze (1990) approach thinking of sense as event. We illustrate this aspect by means of an example. The sense expressed by "Equation (1.1) has two complex roots" might be an event that includes uttering it as part of a problem-process calling for the finding of its roots, and/or the discrimination of whether they are real or complex. A sense of Equation (1.1) is a certain problem-solving event. There are no senses without events. Senses emerge out of the way we deal with a situation calling for an event, such as the determination of the roots of Equation 1.1.

Senses can be abstract in different regards. "Equation (1.1) has two complex roots", for example, could be said to be abstract because, say, the roots cannot be made visible in a regular Cartesian graph. Sometimes an attribution of abstraction is dependent on the experience of problem solvers; for instance, someone conversant with complex functions as $R^2 \rightarrow R^2$ mappings might conclude that complex roots are as visible as real ones. But in other contexts the quality of abstraction may not be so relative to the competencies of problem solvers. Dividing polynomials, for instance, may be deemed abstract, perhaps, because polynomials do not seem to be something that can be fractionally divided, although this conclusion may be dependent on the kind of expectations that the word "divide" tends to elicit. These reflections lead us to highlight the complexity inherent in the quality of abstraction, as it pertains distinct regards with which a given sense is undertaken by the event of its enactment. Attributions of a sense being more or less abstract get distributed along different dimensions. Because of this, there is no basis to choose a path of increasing abstraction to characterise the overall trajectory of mathematics learning. Learning mathematics involves abstraction as a matter of dynamic interlacing of shifting and sustaining relationships of many kinds. To express our vision of these dynamics and, therefore, rethink the attainment of abstraction, we introduce the idea of paths of white and black light.

2. Paths of White and Black light

Concluding his commentaries about multiple mythical narratives, such as the one of Thales measuring the height of an Egyptian pyramid by the shadow of a stick, or the use of the gnomon in ancient Babylonia, Serres (2017) insists: "Yes, its abstraction is a sum and not a subtraction" (p. 210) and introduces the image of white light: "Geometry integrates all our practical or ideal habitats the way white light sums up all the colours, in transparency or translucency" (p. 210). This remark has inspired us to distinguish paths for the realization of abstraction corresponding to white and black light.

Whereas the path of black light is abstraction by means of subtraction of sensible qualities, the path of white light meanders in the midst of a *surplus* of sensible qualities.

To illustrate the difference between abstracting paths of white and black light we invoke Peirce's distinction between a general and an unknown. Let us start with the notion of a general:

> A sign is objectively general, in so far as, leaving its effective interpretation indeterminate, it surrenders to the interpreter the right of completing the determination for himself. "Man is mortal." "What man?" "Any man you like." (CP 5.505)

A theorem proving a property of triangles, for example, deals with triangles as a general. A general is genuinely indeterminate. Note that generals are not necessarily produced by generalizations, since the latter involve no more than extending a finite set of empirical observations. 'Triangles,' as a general, refers to a multiplicity of items *that are all actively related to each other through a continuous and mutual communication of differences*. In fact, Peirce saw a deep connection between generals and the continuum. Peirce deemed that a general is unlike a finite or infinite set of discrete elements. Even the set of real numbers, customarily used to illustrate the continuity of a line, would not correspond to a continuum according to the late Peircean sense, because no matter how many infinite points are added to a set they still remain in isolation from each other. What truly establishes a continuum is a mutual communication or connectedness that cannot arise from isolated elements, regardless of their numerosity. In contrast to generals, Peirce characterised unknowns — particulars with certain but unspecified traits — as "vague":

> A sign is objectively vague, in so far as, leaving its interpretation more or less indeterminate, it reserves for some other possible sign or experience the function of completing the determination. "This month," says the almanac-oracle, "a great event is to happen." "What event?" "Oh, we shall see. The almanac doesn't tell that." The general might be defined as that to which the principle of excluded middle does not apply. A triangle in general is not isosceles nor equilateral; nor is a triangle in general scalene. The vague might be defined as that to which the principle of contradiction does not apply. For it is false neither that an animal (in a vague sense) is male, nor that an animal is female. (CP 5.505)

We are uncertain whether the eye colour of a friend is green or brown, but we know that it is not, say, red. The vagueness of her eye colour includes infinite shades of brown and green and excludes redness. Together with such vague sense of eye colour, we may also presume that her eyes are of a particular colour, which is the key character of an unknown: its traits are determined but we know them only vaguely.

Grappling with an unknown entails relating to an entity that lacks, perhaps only momentarily, certain sensible qualities both in itself (e.g. her eye colour) or in its signs (e.g. a textual description of her eye colour). On the other hand, we navigate a general, such as mortals or triangles, by immersing ourselves in a vast and familiar terrain of sensible variations and differences, such as mortals of different age, sex, species, bodies, and behaviours; or triangles differing in shape, size, angles, perimeters, and colours. The high school problem of determining the length of a side of a triangle, given the length of its other two sides and the angle in between them, is likely to confront us with an unknown, that is, a quality that is only vaguely known (e.g. if two sides are a few centimetres long, the length of the third one is vaguely known to be shorter than a meter). On the other hand, working to demonstrate that, for *any* triangle, the sum of the lengths of two sides is longer than the length of the third side, *whatever* one is chosen as the latter, calls us to deal with a general encompassing an infinite number of triangles, not even countable, displaying distributions of infinite possible qualities; unless we prove it by blindly following a scripted sequence of steps, we are likely to be dazzled by the all-embracing universe of entities we are dealing with. In brief, generals have to do with paths of white light, engaging us with an untold richness of qualities, while unknowns regard paths of black light, namely, darkly illuminated paths along which qualities of interest get to be blurred or somewhat indistinct. Working with generals comprehends all the nuances that pertain to an inexhaustible field we are navigating, so it develops along a path of the white light type, while working with unknowns concerns the subtraction or blurring of qualities, so it relates to a path of the black light type.

A question we strive to address in this study is precisely: *What kind of navigation arrives at abstraction across a surplus of sensible qualities, that is, of the white light type (in terms of generals)?* We examine this question through selected episodes in which children explore the kinaesthetic production of graphical expressions, for a general that can be named by the equation: $A - B = C$. We situate our study within the growing field of early algebra (Kieran, Pang, Schifter, & Fong Ng, 2016). The emphasis of the early algebra work tends to be on the logic of unknowns and on generalising processes with respect to patterns, variables, structures and relational thinking (Blanton et al., 2016; Bodanskii, 1969/1991; Carraher, Schliemann, Brizuela, & Earnest, 2016; Kaput, 2008; Kaput, Blanton, & Moreno, 2008; Ng & Lee, 2009; Radford, 2014). While marginal, generals are also part of the early algebra literature; Davydov (1990), for instance, proposes ideas that seem to engage children with generals: "In many students even by the end of grade 1 and the beginning of grade 2 (8 years) we detected systematic reasoning about rather complex mathematical *relations*, about their *connection*, and all of this was done without objects, on a purely verbal level or by relying on letter formulas." (1990 p. 170 Italics added). The work described in this paper belongs to early algebra, we suggest, because it involves the elementary symbolic treatment of unknowns and generals. We also situate it within the context of a specific family of mathematical instruments that involve kinaesthesia and the use of sensors.

3. Sensors, Kinaesthesia, and Mathematical Instruments

In this paper, we attend to the kinaesthetic production of graphical expressions by means of a mathematical instrument. By "mathematical instrument" we refer to a material implement used interactively by means of individual or collective continuous body movements, to obtain and transform mathematical expressions – differently from the idea of instrument as discussed in the instrumental approach initially introduced by Verillon and Rabardel (1995), because we want to avoid reference to a psychological characterisation of instruments through schemes of usage by the subjects, as it entails a dualism separating mental processes from bodily/material actions (Nemirovsky, Kelton, & Rhodehamel, 2013). We stress *continuous* body motion: a body does not jump from one spatial configuration to another without traversing interconnected trajectories over time. It is the case that some tools driven by body motion produce discrete sequences, such as texts typed on a computer keyboard, so that the intermediate trajectories between key presses are literally ignored. This is not inherent to keyboards: the performance of any experienced piano player shows that body motion in between key presses fully participate in the musical expression.

Classic examples of mathematical instruments are ruler and compass; other examples are instruments to draw curves, such as ellipses or cycloids. A computer mouse is also an instrument, which in the context of certain software environments may count as mathematical as well, as is the case of dynamic geometry in which dragging becomes a key movement for the tracing of geometric properties (e.g. Baccaglini-Frank & Mariotti, 2010; Sinclair & Yurita, 2008; Straesser, 2002) . It has been studied in the literature how the fluent use of a mathematical instrument involves the adoption of a "tool perspective" by the users (Nemirovsky, Tierney, & Wright, 1998). The idea of tool perspective encompasses the emulation of tool's sensitivity to some aspects of activity rather than others, as well as the recognition of conditions and patterns under which a certain tool-use is significant. We will describe instances of a type of learning situation in which children encounter and productively deal with some algebraic generals through the use of a mathematical instrument we have named "WiiGraph" which was designed by a team led by Ricardo Nemirovsky.

Among the many possible settings of WiiGraph, there is one in which the distances between two hand-held remotes (or Wiimotes) and a LED bar are graphed over time, while a third graph, corresponding to the differences between these two distances, is also displayed in real time (Figure 2). The colour of each position vs. time graph corresponds to the colour of the Wiimote being recorded (i.e. light blue and pink; the presence of two large dots with these colours on the screen indicates the sensor as connected to the Wiimotes), or a different one for the case of the difference graph (i.e. dark blue; Figure 1).

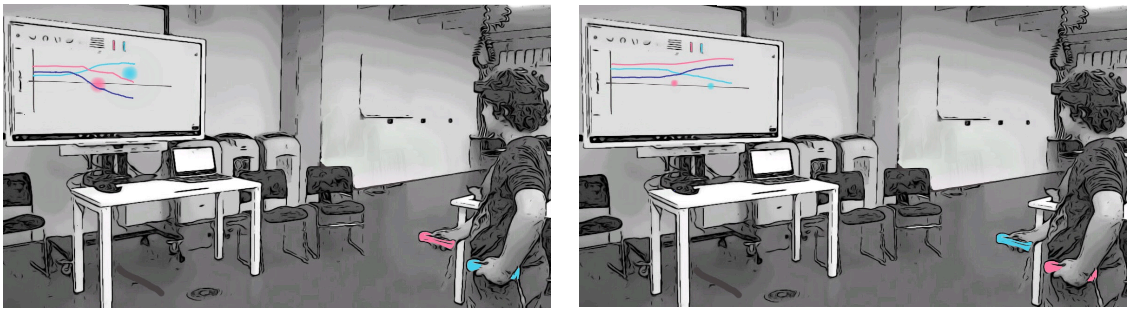

Figure 1. WiiGraph in mode A–B

WiiGraph combines two motion detectors working at body-scale, that is, involving wider body movements than, for example, those required in moving a mouse, like walking in space or overarm gestures. It displays two movements occurring simultaneously, whether performed by one or two people moving at a time, or two hands, providing continuous feedback to the users. The relationality of the two movements can be expressed in different ways, such as through their difference or ratio. Questions steering investigations with WiiGraph may concern the sustaining of a certain value for these relative ratios or differences over time, as the user interacts by means of body movements. Solutions to these questions end up taking the form of kinaesthetic patterns such as walking with a

Wiimote in each hand with a fixed distance between the two Wiimotes. There are two traits of such investigations that we highlight:

1. The distances in question can be genuinely indeterminate: the instruments, recalling Peirce's quotation, "surrender to the interpreter the right of completing the determination," which implies wayfaring investigations that involve generals (wayfaring is used in contrast to the more static idea of transport between two fixed points, its beginning and end. Ingold, 2007). It is by means of wayfaring that new kinaesthetic tasks can emerge out of the explorative activity with WiiGraph.
2. While these generals can be treated as functional relationships (e.g. given one distance what is the value of the other one?), they do not have to be. The symmetry between each pair of distances allows for a relationship in which the distances are not assigned respectively to domain/codomain or input/output, and the attribution of number values is optional. Furthermore, these instruments can be used with arbitrary functional relationships, such as, instead of A–B = 2 the investigation could be about A–B = C with C "surrendered to the interpreter."

Abrahamson and Sánchez-García (2016) studied the use of an instrument that is similar to the WiiGraph with regard to the users' kinaesthetic engagement. The main thesis of their study is that learning involves "moving in new ways." This can be equally valid for the learning of sports, musical instruments, or mathematics. Given that ways of moving that are formative, say, for playing basketball are not necessarily relevant for playing piano or ping-pong, the question arises: What "new ways of moving" would count as formative to algebra learning? In particular, how do these ways of moving come to mind the gap, often pointed out in the literature, between bodily action and symbolic mathematical activity? Kinaesthetic exploration of generals, such as the one corresponding to the equation A – B = C, are our mathematical keys to discuss the ways in which minding the gap may occur through the synergy between mathematical instrument and body motion.

We worked with a group of four children aged 11 years, who did not previously know each other, over three sessions. The children had been recruited as volunteers through a network of families practicing home schooling education. Since they do not attend regular lessons at school, we cannot infer their mathematical background. The participants were filmed with two fixed cameras during each session and two of them wore a head-based Go-Pro camera. In addition, we recorded the computer screen with a computer-generated video that later we synchronized with the video from the cameras. The sessions took place at a classroom of a university in England. The conversations with the children were conducted in English. Several of the parents were present in the classroom. During the first two sessions they explored position vs. time graphs generated by two children, each moving a Wiimote. In addition to free explorations, they engaged in diverse activities anticipating and matching body motions and graphical shapes of position vs. time. In the third session three children worked by holding both the remotes individually, one remote in each hand. As opposed to a pair of children each handling one Wiimote, the one-in-each-hand arrangement differs markedly, among other reasons because of the centrality it confers to relative arm motion (Nemirovsky, Kelton, & Rhodehamel, 2012). The instructor chose to turn on the difference graph, as a significant way of exploring relationships between graphs and body motion, beginning the episodes we examine in the next section. We have selected these episodes because they span the students' production and exploration of the difference graph. The first and the last author were both present in the classroom. Dan, Mario and Zev are the names we use for the children.

The analysis of each episode is synthesized in its ensuing commentary. We focused our analyses on talk, body motion, gesture, and tool-use. The included commentaries elaborate on those aspects that, we felt, led us to insightful remarks. However, for the sake of completeness and to allow for the belike possibility that readers would develop interpretations that did not occur to us, the annotated transcript describes, as far as possible, all the events that took place, including those that did not elicit an explicit commentary from us.

Selected Episodes: Exploring the Difference Graph

Segment 1: Introducing the difference graph and trying to keep it on zero

1. Ricardo: The computer also generates another line [turns on the difference graph] that is, em, dark blue [points at the dark blue graph; Figure 2a] (…) so we'll investigate what this third line is doing there, what it's showing. So, the first thing we'll try…

2. Mario: It's called, it's called minus because that, that purple [dark blue] line, line is, is, is pink minus blue.

3. Ricardo: OK, how do you know that?

4. Mario: It's real, it's quite obvious, where it says pink minus blue [points to the screen, note the area pointed at with a black arrow in Figure 2a] at the top of the screen.

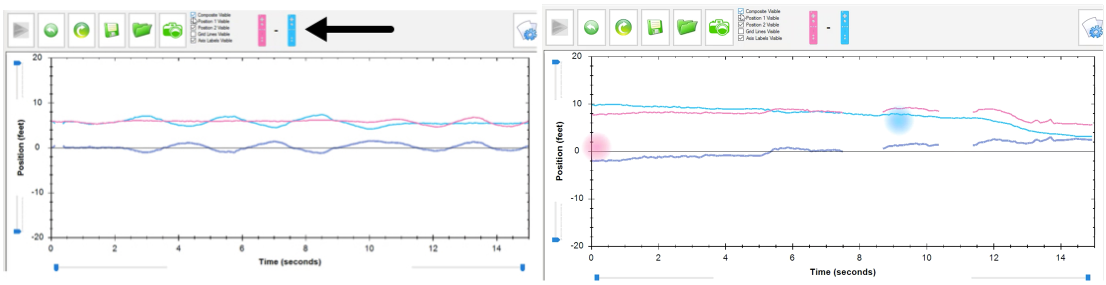

Figure 2: (a) Graphical display in which the dark blue difference graph is displayed for the first time; (b) Mario's first attempt to create a difference graph

5. Ricardo: Aha (…) [gives the two remotes to Mario] So you move, you do whatever you want, [moves alternately right and left hands] but try to keep the dark blue on zero [points to the dark blue line], on this line [left hand runs along the *x*-axis].

Mario begins his first difference graph: he starts with the pink remote in his left hand and the blue one in his right hand. At the beginning of the experiment, the pink remote is kept slightly ahead of the blue one, and then the two are slowly switched in their positions. Holding the two remotes separated, he then walks forward (see the graphs in Figure 2b).

During the last seconds of the graph production, he separates the remotes even more and says:

6. Mario: I'm trying as hard as possible not to make the things go opposite.
7. Ricardo: So here you, this piece [pointing to the dark blue graph around second 7] you had it on the line… so try to do more of that, see if you can.

While Ricardo is speaking, Mario moves both remotes back and forth, swinging the arms rhythmically. Then, he starts a new session, moving the remotes slowly in opposite directions, and produces the graphs in Figure 3a:

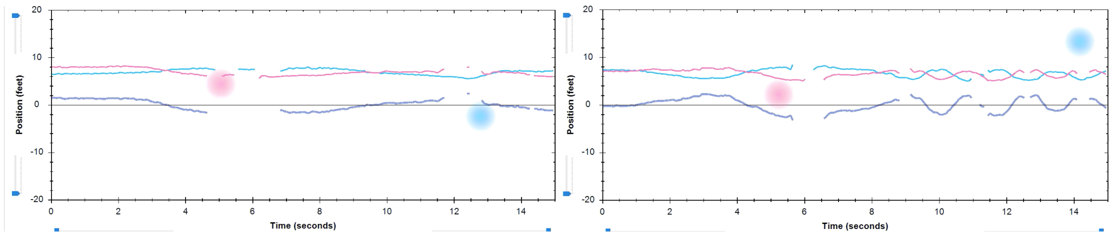

Figure 3: (a) Mario's second attempt to create a difference graph; (b) Mario's new session

During the last seconds he says:

8. Mario: They're both neutralising each other.
9. Ricardo: Uhum.
10. Mario: That's because, because most of the time I'm, I'm, pink's going in a straight line and blue's going in a stripe, straight line ((inaudible)) [stops talking, while a new triplet of graphs starts to be created superimposed to the previous one; moves the remotes again back and forth].

Mario presses a button in the Wiimote and a new session starts: he alternates fluid back and forth movement of the two remotes, which becomes faster and faster (Figure 3b).

11. Ricardo: So here they were these lines [points to the initial part of the dark blue line in Figure 3b]… they, oh, look! [points to the intersection of the dark blue line with the time axis, as the lines unfold] [after second 8, Mario starts to rhythmically bounce on his knees while accelerating the fluid back and forth arm movement creating the second half of the graphs in Figure 3b]

Commentary

The appearance of a third graph prompted Mario to examine the screen seeking for additional signs that could account for it. There was none with a dark blue colour. However, the sign at the top of the screen "pink minus blue," which had been displayed from the beginning of this session but had remained unused, offered him a compelling interpretation ("it's obvious"): the dark blue line "is called minus." The inscription above the graph included the pink and blue Wiimotes, freeing the minus to be clasped by the third graph. The dark blue graph seemed to announce its name. In Paragraph [2] Mario expressed an initial sense for the dark blue graph focused on its name.

Mario started the graph shown in Figure 2b with the pink Wiimote in front of the blue one, slowly moving them towards their centre. Once they were next to each other he continued slowly moving them along the same directions. Right before 8 seconds the pink graph disappeared, possibly because the orientation of the pink Wiimote made it fall outside the field of reception. This interruption is likely to have prompted Mario to move the pink Wiimote to make its graph reappear. When it did, the dark blue graph was above the *x*-axis. Then his arms tensed as if trying to push the dark blue graph toward the *x*-axis. Mario reflected on this sense of effort ("trying as hard as possible") as striving "not to make the things go opposite" (Paragraph [6]). This "going opposite" (an event, in Deleuzean sense) might have been the dark blue graph moving in a direction opposite to the desired one, such as toward the zero line. Another possibility, evoked by Mario's use of the plural "things," is that he saw the pink and light blue lines moving in opposite directions, instead of, perhaps, staying together. His reaction was to try to "lower" the pink and blue graphs by walking toward the monitor. However, the dark blue graph continued to inch upwards.

In all his graphical productions (Figures 2b, 3a and 3b) Mario tended to move the Wiimotes in alternate directions. This is likely to have followed from tacitly adopting Ricardo's demonstration (see Paragraph [5]). Ricardo gestured an alternate movement of the Wiimotes while saying "you do whatever you want" (Paragraph [5]). While words may leave to the interpretant a more open range of possibilities, gestures are inclined to convey unintended specificities. This tacit assumption of a wavy kinaesthetic pattern was in tension with the task of maintaining the dark blue graph on the *x*-axis. The graph "called minus" was not just a visual display out there, but also a curve that resisted physical efforts seeming to possess a will of its own which at times led Mario to tense his movements.

In Paragraph [8] Mario expressed a sense for a general relationship between the pink and blue graphs: "They're both neutralising each other." While his ensuing account of this relationship in Paragraph [10] is inaudible, we hear the sense of Paragraph [8] as indicating the emergence of a general. Recall that dynamic relationships between components affecting each other constitute generals: "neutralising" suggests a present continuous activity interrelating two graphs or Wiimotes.

Around second 8 of the graphs shown by Figure 3b, Mario seemed to free himself from trying to keep the dark blue graph close to the horizontal axis, engaging in a new rhythmic kinaesthetic pattern swinging his arms back and forth and bouncing his knees. This bodily movement expressed itself visually by a wavy synchronised variation of the three graphs at once. Relieved from trying to push the dark blue line horizontally, Mario seemed to enjoy a relaxed and smooth swinging — a sense for the graphical weaving emerging on the computer screen as expressed by his wavy body motion.

Mario gives the Wiimotes to Dan, who starts a new graph. He stands still in the same position for all the session, keeping steadily the remotes at the same distance from the sensor (Figure 4a).

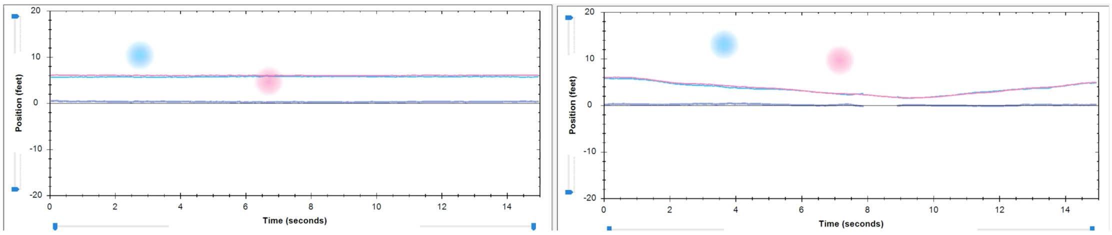

Figure 4: (a) Dan generates a graph staying still with the Wiimotes next to each other; (b) Dan keeps the difference graph on zero while walking

12. Ricardo: So that, that's a perfect zero! [around the 8^{th} second, laughs] [ending his graph, Dan relaxes his position, shrugs his shoulder and smiles]

13. Ricardo: And, can you do it while you walk?

Dan starts moving very slowly towards the sensor with both the remotes kept steady and then backwards; he generates the graph shown in Figure 4b:

14. Dan: You just have to keep the remotes in (…) one position.

15. Ricardo: Like, keeping [them] together?

16. Dan: Keeping them at the same level.

17. Ricardo: The same level, ok.

Dan creates then new graphs, walking again towards the sensor, then backwards, keeping the remotes steady, next to each other.

Commentary

Dan came to create a horizontal difference graph with a clear plan — stay still with the two remotes next to each other — that he had developed while observing Mario's experimentation. He had a well-defined sense that a dark blue graph on the horizontal axis "converted" into the two Wiimotes being next to each other. Moreover, Dan easily showed in Figure 4b that that condition was indifferent to his walking distance from the sensor: "You just have to keep the remotes in one position" (Paragraph [14]). Dan articulated his sense for the relationship between the dark blue graph being on the *x*-axis and the range of kinaesthetic activities consistent with it in two ways: "keep the remotes in (…) one position" (Paragraph [14]) and: "Keeping them at the same level" (Paragraph [16]). While the word "position" alludes to a location in space, the word "level" is customarily a term for height. So far, the children's experimentation with the Wiimotes had not included varying the kinaesthetic quality of the Wiimotes/hands' height, to ascertain graphical responsiveness. On the other hand, differences in height between the light blue and pink graphs had been of major significance. We surmise that Dan's relevance of the Wiimotes being at equal levels had drifted from noticeable graphical levels to the taken-by-default levels of the Wiimotes. We will call this "sliding" of qualities of one signifier (e.g. graphs' levels) for another (e.g. Wiimotes' levels), which end up encompassing both, with the term 'semiosis' that Peirce introduced to mean "an action, or influence, which is, or involves, a cooperation of three subjects, such as a sign, its object, and its interpretant" (CP 5.484). We conjecture that the migration of qualities between signifier (i.e. sign) and signified (i.e. referred object) is the crucial event (i.e. "action or influence") taking place in the continuum that Peirce has called "interpretant." For instance, for an English speaker the word "smooth" sounds smooth, as if the sound had borrowed such quality from the smooth entities it qualifies. The same for the word "sharp." The interpretant, then, is a continuum sustaining the migration of qualities — their expansion, contraction and reproduction across signifieds and signifiers.

We propose that semiosis is a key process in the formation of a general, which reflects the inherent presence of a continuum across which the elements of a general communicate with each other. The graphs on the computer screen intermingle with the hand-held Wiimotes, such that "same level" and "same position" can refer to all of them and distribute qualities that are distinct and yet, overlapping.

Dan and Zev exchange the remotes. As soon as Zev grabs the remotes, he starts a new session and creates the graphs in Figure 5:

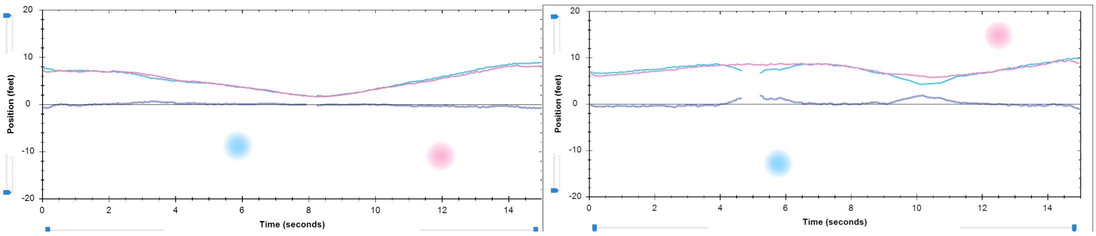

Figure 5: Zev keeps the difference graph on zero while walking

18. Ricardo: So what do you think, how do you explain?
19. Zev: Well, every walk I've done checks, em… they've got a descending number and that's the distance of each control on the sensor and then minuses the red one from the blue one [points towards the two remotes depicted on the top of the screen]. So if they're both the same [keeps the remotes next to each other], one minus one is zero, and the same with two minus two, so when we move them back and forwards the same [moves again both the remotes in a coordinate manner back and forth and starts a new session, beginning Figure 6] it stays at zero, but when we move one [moves one remote backwards while he keeps the other one still, see the region around the arrows in Figure 6].

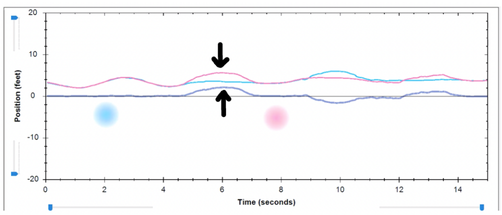

Figure 6: Zev creates a new difference graph while speaking

20. Ricardo: So here, this minus this distance is zero [points to two overlapped points of the blue and pink lines, then to their difference graph] … But here, what did it happen? [points to the two points of the pink and dark blue graphs marked by the arrows in Figure 6]
21. Zev: Well, it's, it's different.

22. Ricardo: It's different… Alright, very good.

Commentary

Zev begins Paragraph 19 by articulating three propositions: 1) "they've got a descending number," 2) "that's the distance of each control on the sensor," 3) "minuses the red one from the blue one." We will comment on the sense of each one and how they concatenate:

- they've got a descending number

"They" are, for the most part, the blue and pink graphs. Furthermore, since Zev had kept the two graphs going together, his saying "*a* descending number" (singular) might suggest that both graphs descended by *the same* numbers. However, semiosis allows "they" to also relate to the hand held Wiimotes, their icons on the computer screen, or the numbers implicit in the shape of the graphs. In the event, the numbers are descending if the corresponding remote gets closer to the sensor. While such getting closer is an action undertaken by Zev, this proposition is articulated from the point of "them," so that they "got" a descending number. In other words, the descent is an effect passively undergone by the graph/number. The subsequent "and" signals the beginning of another proposition in the form of a juxtaposition, that is, the upcoming proposition is to be held in parallel with the previous one.

- that's the distance of each control on the sensor

"That" brings up from the prior proposition a descending number to predicate of it that it is a specific distance between Wiimote and sensor. This specification is inscribed in the general whose sense Zev is articulating: a general in which descending numbers, walking towards the sensor, lowering graphs, and distances between Wiimotes and sensors are all mutually conditioned. The subsequent "and *then*" betokens an upcoming proposition that is not so much to be juxtaposed as coming after the prior ones.

- then minuses the red one from the blue one

After the numbers are gotten, they are "minused." Zev states that the red one is minused from the blue one. This can be understood in opposition to the equation depicted above the graphical space that appears as if blue is to be "minused" from red. However, the object of Zev's explanation is the case of red and blue numbers being equal, so that the result, zero, is indifferent with respect to which is minused from which.

In "one minus one is zero, and the same with two minus two" Zev uses particular examples to illustrate a general relationship. This is an instance of what Mason and Pimm (1984) have called "seeing the general in the particular." Zev is articulating a general that we could symbolize by: $A - A = 0$; however, his understanding is far from being reducible to any formal definition: it encompasses countless qualities, such as the kinaesthesia of walking with two hands next to each other, the light blue and pink graphs going at the same height, the dark blue graph staying over the horizontal axes, the vast number of numbers that can be subtracted from themselves, the nothingness that remains after taking away — minusing — what had been given, or walking ahead as a kind of "descending." Navigating such boundless expansion of interrelated qualities is what we characterize as a path of the white light, which gives rise to a general. Zev says that this general encompasses "when we move them back and forwards *the same,*" then he begins to point out, in words and gestures, that it excludes the case "when we move one…" and not the other one. This inclusion/exclusion criterion for the general in question is reaffirmed by Zev ("it's different", Paragraph 21), as he qualified the two cases pointed out by Ricardo in Paragraph 20.

Segment 2: Keeping the difference graph above or below the *x*-axis

23. Ricardo: So now we'll try to do something similar but keep the dark blue line always above. You can walk and move your hands, but keep the black, the dark blue line above the zero.

Mario comes to the front and holds the remotes. He starts a new session (Figure 7a):

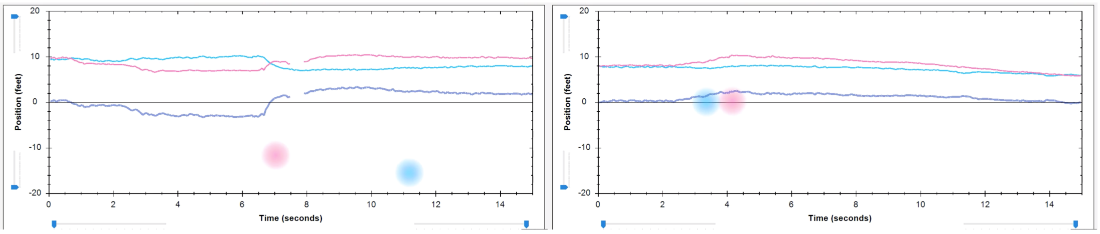

Figure 7: (a) Mario creates a new difference graph; (b) he creates a difference graph above the *x*-axis

24. Ricardo: So now it's above [points to the dark blue graphs, around 10 seconds].

Mario creates a new graph (Figure 7b):

25. Ricardo: So what do you think? In order to get the dark blue line above the zero, what do you have to do?
26. Mario: Well, you, you make pi, pink bigger than blue so that…
27. Ricardo: … the pink…
28. Mario: … so you keep it above but if you wanted it below you have to have blue bigger than pink.
29. Ricardo: Ok, so let's have it, let's have it below now.
30. Mario: What?
31. Ricardo: Let's have it, the dark line, below.

Mario starts a new session but very soon he presses a button that generates again superimposed graphs. Refreshing the screen, Mario creates new graphs with a new session, starting with the pink remote in front, while the light blue one is kept farther from the sensor (Figure 8):

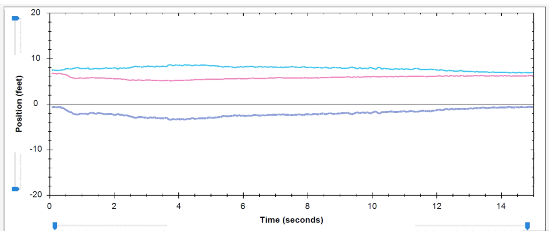

Figure 8: Mario creates a new difference graph under the *x*-axis

32. Ricardo: … [while Mario is moving the light blue remote closer to the pink one] and then slowly you get it to zero. So what did you do to, to have it under the zero line?
33. Mario: I had to make the blue bigger than pink.
34. Ricardo: Ah!
35. Ricardo: So let's, eh, hand it to Dan, and so you create a pattern, you can do as many variations as you want, like walking and moving, but always keeping the blue line above the zero to start with…

Dan starts a new difference graph (Figure 9a)

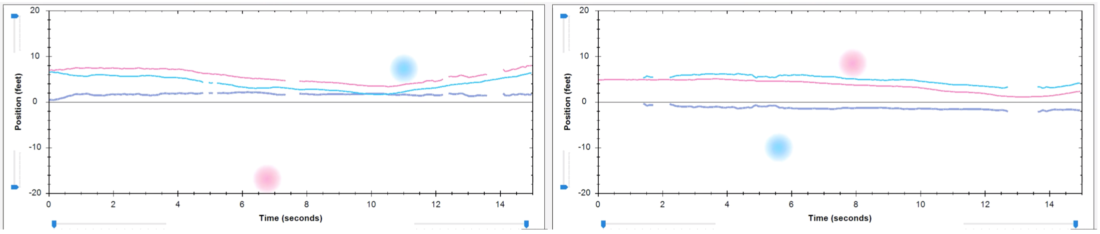

Figure 9: (a) Dan creates a new difference graph above the *x*-axis; (b) Dan creates a new difference graph under the *x*-axis

36. Ricardo: So what did you do, to keep it above?
37. Dan: Em, pink back and blue forward.
38. Ricardo: The pink back, further from this [pointing to the sensor bar]. Em, and now keep it under this [the *x*-axis].
39. Dan: Oh! [restarts the session several times while Ricardo makes a few comments]

Dan creates the difference of Figure 9b.

40. Ricardo: So what, what does it happen here, to get it below [the *x*-axis]?
41. Dan: You've to do the opposite… you put pink forward and the blue back.
42. Ricardo: Pink below, right? Ok. Great!
43. Ricardo: So now, Zev, do, do something like this: above, below. So first of all, but try to find out the variation, so what is it possible?

Zev is given the remotes and creates the graphs of Figure 10a:

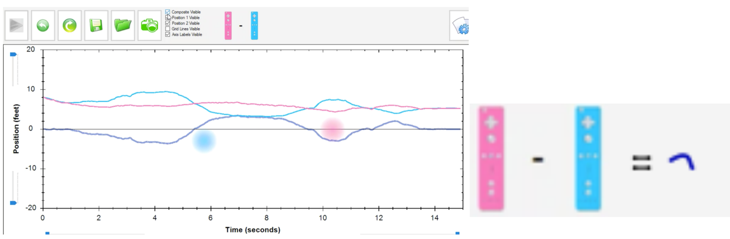

Figure 10: (a) Zev generates a graph in which the difference graphs goes above and below zero; (b) possible representation of the equation Zev discusses in Paragraph [49]

44. Ricardo: So, how did you change [the dark blue line] from below to above?

45. Zev: Em, by changing which controller was in front.

46. Ricardo: So which one was in front here? [points to the dark blue graph around the 4th second, where it is below the *x*-axis]

47. Zev: Em, [light] blue.

48. Ricardo: (…) And do you have a sense for why for the blue, for the dark blue line, to be below [the *x*-axis] then the pink has to be below [the light blue graph]?

49. Zev: Em, yep. Em, it's something to do with like maths and, like, because on there, it says the [seeming to point at image of the two Wiimotes on the screen] has been taken away and then it's hard to tell because it's not actual numbers, but, if you have more on one side, that will be a negative number… then, then, if you have them on the other side, it'll be a positive number, which is that [moves the dots on the screen along the dark blue line by controlling the remote].

Commentary

In Segment 2 Mario, Dan, and Zev characterized two complementary regions, one for the dark blue graph being above the *x*-axis and another for it being below. Mario separated these regions by contrasting "pink bigger than blue" and "blue bigger than pink" (Paragraphs [26] and [28]); Dan by "pink back and blue forward" or "pink forward and the blue back" (Paragraphs [37] and [41]); and Zev "by changing which controller was in front" (Paragraph [45]). In our second commentary for Segment 1, we elaborated on a notion of semiosis centred on the migration of qualities among signifieds and signifiers — a migration or drift hosted by the interpretant. We also suggested the image of the interpretant as a continuum sustaining the "expansion, contraction and reproduction [of qualities] across signifieds and signifiers." Segment 2 inspires us to visualize compositional elements for the interpretant; namely, that instead of a single all-encompassing continuum, the interpretant would be akin to a Riemann surface with various sheets. (see http://mathworld.wolfram.com/RiemannSurface.html) Mario, Dan and Zev suggested several sheets: 1) a sheet to host the dark blue graph with a region above the *x*-axis, distinct from a region below the *x*-axis, 2) a sheet hosting the pink and light blue graphs with regions separating which one is "bigger," 3) a sheet containing the Wiimotes distributed along regions demarcating which one is closest or farthest from the LED bar, and 4) a sheet containing the Wiimotes distributed along regions that distinguish which one is in front of which one. The regions of each sheet map out with regions on the other sheets, such that, for example, "pink bigger than blue" in one sheet maps out with the "dark blue graph above *x*-axis" in another sheet. The children collectively unfolded semiosis as hosted by an expanding interpretant with multiple sheets, allowing for the mutual discrimination of diverse qualities, such as Above/Below, Bigger/Smaller, Back/Forward, and Front/Behind.

In the Introduction section we proposed that abstracting along a path of white light involves dealing with a surplus of sensible qualities, many of them actively related to each other through a continuous and mutual communication of differences. In this commentary we suggest that these active relationships and mutual communication of differences take place across an evolving manifold that we identify with Peirce’s interpretant.

In Paragraph [49] Zev explained how the distinction between regions in some of these sheets relates to numbers. First, he points the remotes drawn above the graphical region indicating that it “has been taken away,” alluding to the subtraction indicated by the Wiimotes above the graphical space. Zev thinks that “it is hard to tell” what happens because the actual numbers are not shown. This latest remark makes a fleeting allusion to an unknown: beyond estimating possible values, the position numbers *per se* are not given. However, Zev quickly leaves behind these unknowns to distinguish two regions by their corresponding “sides:” more on one side of the minus will obtain a negative number, more on the other side a positive number. Zev described here a fifth sheet, which included the two Wiimotes depicted on the computer screen separated by a minus, with left and right sides or regions, such that having “more” on each side maps out numbers for the dark blue graph being positive or negative.

Discussion

During Segments 1 and 2, Mario, Dan and Zev experimented with WiiGraph to make sense of a new graph — coloured dark blue — appearing on the computer screen. Through these experimentations, the children strived to attend and interpret a vast scope of sensible qualities, including not only all that was “there,” such as shapes of three graphs mutually distinguished by colour, degrees of closeness to the monitor, moving dots that had to be kept inside the screen, signs of subtraction, unknown numbers, relative heights of graphical lines, a horizontal line sometimes called “zero,” pink and blue Wiimotes with buttons on them, video cameras, synchronic movements of graphical lines from left to right, and more, but also all the kinaesthetic and proprioceptive qualities realized in the course of their movements and actions. The latter included the tonicity of muscular activity, moving hands while standing still, walking while keeping hands still, walking and moving hands cyclically, keeping hands next to each other, keeping one hand closer to the LED bar than the other one, bodily-kinaesthetic responsiveness to events on the computer screen, changes on the computer screen responsive to their movements, going fast and slow, moving smoothly and abruptly, and more. Such abundance is what we refer to as a “surplus of sensible qualities.” Over time, some of these qualities became more peripheral or more central than others. It was by virtue of kinaesthetic engagement that the graphs developed a temporality marked by complex and all-encompassing events, such as the graphs “going opposite” (Paragraph [6]), creating a “perfect zero” (Paragraph [12]), or getting “a descending number” (Paragraph [19]). Through kinaesthesia, signifiers and signifieds such as “level” and “position” (Paragraphs [14]-[17]), “forward/back” (Paragraph [41]) and “above/below” (Paragraph [45]), exchanged qualities by participating in the ongoing semiosis.

We think that navigating a surplus of sensible qualities along paths of white light is a condition for the encounter and familiarization with a general, which in the present case study is one that can be symbolized by A – B = C. *What kind of navigation arrives at abstraction across a surplus of sensible qualities, that is, of the white light type (in terms of generals)?* is the main question that we tried to address in this paper. Through the analysis of talk, gesture, and tool-use, we came to two intertwined processes appearing to characterize such navigation: semiosis and inhabiting a growing interpretant. The most central aspect of semiosis, we propose, is the exchange of qualities among signifiers and signifieds (e.g. descending number ↔ moving closer to the LED bar, Paragraph [19]). In semiosis qualities move across a continuum (i.e. the interpretant) hosting signifieds and signifiers. We characterized this continuum as a Riemann surface with multiple sheets, each of which harbours regions (e.g. blue graph above or below the zero line) in mutual correspondence with others (e.g. pink Wiimote ahead or behind blue Wiimote). We summarize then, by saying that abstraction along a path of white light entails familiarity with a layered continuous interpretant enabling the active exchange of sensible qualities while keeping them distinct and communicating. It is through this dynamic exchange, which involves proprioceptive and kinaesthetic activity with the instrument, that the encounter with 'A – B = C' occurs and that activity becomes relevant in approaching early algebra, as a way of mobilising variables and equations through the body while opening up a range of possibilities for semiosis. Further research on body motion, the use of mathematical instruments, and the interweaving of generals and unknowns, is vital for the enrichment of early algebra.

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